Differential Geometry

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Contents

Submanifolds

We first consider an m-dimensional submanifold of $\mathbb{R}^n$ (m<n).

Definition: *A continuous/differentiable m-dimensional submanifold of $\mathbb{R}^n$ is a subset $K \subseteq \mathbb{R}^n$ such that*

- $\forall p \in K \quad \exists r > 0$ *such that* $K \cap B_r(p)$ *is the graph*
$$x_i = f_i(x_1, ..., x_m), \quad i = m+1, ..., n$$
- f_i *are all continuous/differentiable*

The $n - m$ remaining coordinates are functions of the first m coordination.

I want to say that there is an EQUIVILENT way to think of this as just a manifold in $\mathbb{R}^n$; I mean a subset is a submanifold if I can write down an atlas for it (using the subspace topology).
Write down a proof

Topological Manifolds

Definition: *A topological manifold $\mathcal{M}$ is a metric topological space such that there exists some $n \in \mathbb{N}$ such that for all $p \in \mathcal{M}$ there is a neighbourhood $\mathcal{U}$ homeomorphic to $\mathbb{R}^n$*

$$\mathcal{M} \text{ is a topological manifold} \iff (\exists n \in \mathbb{N})(\forall p \in \mathcal{M})(\exists \mathcal{U} \in \mathcal{T}_{\mathcal{M}})(\exists \phi : \mathcal{U} \to \mathbb{R}^n \text{ a homeomorphism })$$

Given that a topological space has a homeomorphism from an open set around every point into $\mathbb{R}$n can we then infer that it is hausdorff, and therefore a topological manifold. I.e. is it a real condition that it is Hausdorff, (metrisable is what Volker said but I want to think about Hausdorff). Moreover the definiton on wiki does not mention metrisable or Hausdorff.
It is a real condition. What I am talking about is a locally Euclidean space and these are distinct from manifolds.

Differentiable Manifolds

The Definition of a topological manifold gives us a cover $\mathcal{A} = \{\mathcal{U}_\alpha : \alpha \in I\}$ as well as a homeomorphism, ϕ_α, for each $\mathcal{U}_\alpha$.

Definition: *The collection $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\alpha) : \alpha \in I\}$ is called an atlas, and each element $(\mathcal{U}_\alpha, \phi_\alpha)$ is called a chart. The open set $\mathcal{U}_\alpha$ is called the domain of the chart*

Consider a pair of indecies α, β such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. The restriction of a homeomorphism is itself a homeomorphism onto its image, thus we know that

$$\phi_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \mathcal{V}_{\alpha,\beta}$$

is a homeomorphism, likewise for

$$\phi_\beta|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \mathcal{V}_{\beta,\alpha}$$

Where both $\mathcal{V}_{\alpha,\beta}, \mathcal{V}_{\beta,\alpha} \subseteq \mathbb{R}^n$. Therefore

$$\phi_{\alpha,\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \mathcal{V}_{\alpha,\beta} \to \mathcal{V}_{\beta,\alpha}$$

is a homeomorphism of R^n. Note that $(\phi_{\alpha,\beta})^{-1} = \phi_{\beta,\alpha}$ and so is also continuously differentiable.

Definition: *If the maps $\phi_{\alpha,\beta}$ are continuously differentiable (as maps from $\mathbb{R}^n \to \mathbb{R}^n$) then we call the pair $(\mathcal{M}, \mathcal{A}) = \mathcal{M}_{\mathcal{A}}$ of the topological manifold and an atlas a differentiable manifold.*

Diffeomorphisms

Let $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$ be two differentiable manifolds with $dim(\mathcal{M}_{\mathcal{A}}) = m, dim(\mathcal{N}_{\mathcal{B}}) = n$ and respective atlases $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in I}, \mathcal{B} = \{(\mathcal{V}_\beta, \psi_\beta)\}_{\beta \in J}$. Now consider a continuous map $f : \mathcal{M}_{\mathcal{A}} \to \mathcal{N}_{\mathcal{B}}$.

Definition: *f is continuously differentiable if $\forall(\alpha, \beta) \in I \times J$ the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ as above is continuously differentiable as a map from $\mathbb{R}^m \to \mathbb{R}^n$*

Definition: *A diffeomorphism of two differentiable manifolds, $f : \mathcal{M} \to \mathcal{N}$, is a homeomorphism such that both f, f^{-1} are continuously differentiable maps.*

Two differentiable manifolds are diffeomorphic if such a diffeomorphism exists.

Compatibility

Consider the following two atlases $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in I}, \mathcal{B} = \{(\mathcal{V}_\beta, \psi_\beta)\}_{\beta \in J}$ on the same topological manifold $\mathcal{M}$.

Definition: *The differentiable manifolds $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{B}}$ are compatible if the identity map*

$$id : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{B}}, \quad p \mapsto p$$

is a diffeomorphism (if it is continuously differentiable)

Lemma. *Two atlases are compatible iff their union is also an atlas.*

Proof. Follows from the Definition

Compatibility is an equivilence relation.

Definition: *For a given atlas $\mathcal{A}$ we can take the union over all compatible atlases, which is itself an atlas (for the relevant space). We call this the maximal atlas induce by $\mathcal{A}$. It is also called the differentiable structure induced by $\mathcal{A}$*

Submanifolds again

Theorem. *Euclidean submanifolds are differentiable manifolds*

Proof. First let $\mathcal{M}$ be the differentiable submanifold. Then $\forall p \in \mathcal{M}\ \exists \mathcal{U}_p$ such that $x_\alpha = f_\alpha(x_1, ..., x_m)$ for all elements of $\mathcal{U}_p$.

Take the neighbourhoods that are supplied by the Definition above $\mathcal{U}_p$ with the following maps

$$\phi_p(q) = (q_1, ..., q_m)$$

the first m coordinates of q with respect to the local coordinates in $\mathcal{U}_p$

Let $\{(\mathcal{U}_p, \phi_p) : p \in \mathcal{M}\} = \mathcal{A}$. We now aim to prove that $\mathcal{A}$ is an atlas for $\mathcal{M}$.

First the becuase the $\mathcal{U}_p$ are open subsets of $\mathbb{R}^n$ we get immediately that the ϕ_p are homeomorphisms onto an open subset of $\mathbb{R}^n$ which is itself homeomorphic to $\mathbb{R}^n$

This is not true, you need the open sets to be convex, in particular we should take balls here.

Next we take $p, p' \in \mathcal{M}$ such that $\mathcal{U}_p \cap \mathcal{U}_{p'} \neq \emptyset$. We have the natural function $g = \phi_{p'} \circ \phi_p^{-1}$ defined on the image under ϕ_p of the intersection of our two open sets.

Let $q \in \mathcal{U}_p \cap \mathcal{U}_{p'}$. Then q's representation in the axis given by both the open neighbourhoods differs only by a rotation and a translation. In particular

$$q_i' = \sum_{j=1}^{n} O_{ij} q_j + c_i; \quad i = 1, ..., n$$

For $O \in SO(n)$ (a rotation). But because the coordianates of $m+1$ to n are a function of the first m we have a representation of g as

$$g(q_i) = q_i' = \sum_{j=1}^{m} O_{ij} q_j + \sum_{j=m+1}^{n} O_{ij} f_j(x_1, ..., x_m) + c_i; \quad i = 1, ..., m$$

Now the differentiablility of g follows from the differentiability (by assumption) of each of the f_j.

Thus $\mathcal{A}$ is an atlas and $\mathcal{M}$ is a differentiable manifold.

Quotient Spaces

If we quotient a topological manifold by some equivalence relation we are not guaranteed to get a manifold.

Definition: *A discrete group of diffeomorphisms is a countable set of diffeomorphism of $\mathcal{M}$ into itself which form a group under composition.*

Theorem. *Let $\mathcal{M}$ be a topological/differentiable manifold and G be a discrete group of diffeomorphisms.*

$\forall p \in \mathcal{M} \exists \mathcal{U}$ *open such that* $p \in \mathcal{U}$ *and* $\forall f \in G[f(\mathcal{U}) \cap \mathcal{U} \neq \emptyset \implies f = id]$
$\implies \mathcal{M}/G$ *is a topological/differentiable manifold*

Tangent Vectors

Euclidean Space

Consider a submanifold of $\mathbb{R}^n$, $\mathcal{M}$ of dimension m

Definition: *The tangent space to $\mathcal{M}$ at a point $p \in \mathcal{M}$is*

$$T_p\mathcal{M} = \{X \in \mathbb{R}^n \text{ attached to } p | X \cdot g_\alpha = 0, \ \alpha = m+1, ..., n\}$$

Attached to p? I get the idea I guess but formalism

Where $g_\alpha = x_\alpha - f_\alpha(x_1, ..., x_m)$ and

$$X \cdot f = \lim_{t \to 0} \frac{f(p + tX) - f(p)}{t}$$

Curves

Definition: *A continuously differentiable parametrized curve γ through $p \in \mathcal{M}$ is a continuously differentiable map $\gamma : [0,1] \to \mathcal{M}$ and $\gamma(0) = p$*

Smooth Function Space

$f : \mathcal{M} \to \mathbb{R}$ is a member of $C^\infty(\mathcal{M})$ right?

The space of smooth (infinitely differentiable) maps on a manifold, $C^\infty(\mathcal{M})$ forms a commutative ring (and linear space) under the following operations: $\forall q \in \mathcal{M},\ \forall f, g \in C^\infty(\mathcal{M}),\ \forall \alpha \in \mathbb{R}$

- $(f + g)(q) = f(q) + g(q)$
- $(\alpha f)(q) = \alpha f(q)$
- $(fg)(q) = f(q)g(q)$

General Tangent Vector

Definition: *A tangent vector v at $p \in \mathcal{M}$ is a real valued linear function on $C^\infty(\mathcal{M})$, $v : C^\infty(\mathcal{M}) \to \mathbb{R}$, such that $\forall f, g \in C^\infty(\mathcal{M})$ the Leibniz rule is satisfied*

$$v(fg) = g(p)v(f) + f(p)v(g)$$

This construction satisfies some basic properties

Lemma.

$$f \text{ is a constant function} \implies v(f) = 0$$

Proof. $f = 0$ case is immediate so we may assume that $f \neq 0$. Let $f = \alpha(-)$ the map that sends $x \mapsto \alpha \in \mathbb{R}$.

$$v(f) = v(\alpha(-)) = \alpha v(1(-)) = \alpha v(1(-)1(-)) = 2\alpha(1(p)v(1(-))) = 2\alpha v(1(-))$$

$$\implies v(\alpha(-)) = 2v(\alpha(-))$$

$$\implies v(\alpha(-)) = 0$$

Lemma.

$$f \text{ vanishes in a neighbourhood of } p \implies v(f) = 0$$

Proof. f vanishes on $\mathcal{M}$ in a neighbourhood of p. Call this neighbourhood W and take a chart that contains p, (U, ϕ). Then $p \in U \cap W$ and f vanishes on $U \cap W$.

We know that $\phi(p) \in \phi(U \cap W)$ which is open because ϕ is a homeomorphism. We can assume that $\phi(p) = 0$ (by merely translating the image, we will still have a homeomorphism). Again becuase $\phi(U \cap W)$ is open there exists an ϵ such that $\phi(p) = 0 \in B_\epsilon(0) \subseteq \phi(U \cap W)$.

Recalling that we have bump functions on $\mathbb{R}^n$, denote one $\rho_\epsilon(x) = \rho(\frac{x}{\epsilon})$.

Then we can define the following smooth function on $\mathcal{M}$

$$g(q) = \begin{cases} (\rho_\epsilon \circ \phi)(q), & q \in U \cap W \\ 0, & \text{else} \end{cases}$$

Because we assigned $\phi(p) = 0$ and ρ is a bump function we get that $g(p) = 1$.

Now the product $fg = 0$ because f is zero on W and g is zero in the compliment of $U \cap W$ (in particular if g is nonzero it is in W). Thus

$$0 = v(0) = v(fg) = f(p)v(g) + g(p)v(f) = 0 \cdot v(g) + v(f) = v(f)$$

These properties together imply that $v(f) = 0$ when f is constant in a neighbourhood of p. i.e. v(f) only sees the local behaviour of f.

Tangent Space

Definition: *The tangent space at $p \in \mathcal{M}$ is*

$$T_p\mathcal{M} = \{v \in (C^\infty(\mathcal{M}))^* : v \text{ satisfies the leibniz rule } \}$$

Theorem. *$T_p\mathcal{M}$ is a vector space with the same dimension as $\mathcal{M}$.*

Proof.
It will suffice to prove that for any $v \in T_p\mathcal{M}$

$$v = \sum_{i=1}^{dim\mathcal{M}} v_i \frac{\partial}{\partial x_i}|_p$$

(the partial derivative form a basis) where $\frac{\partial}{\partial x_i}|_p$ is notation for the functional

$$(\frac{\partial}{\partial x_i}|_p)(f) = \frac{\partial}{\partial x_i}(f \circ \phi^{-1})|_0$$

Where ϕ is the homeomorphism from the chosen chart of $\mathcal{M}$ containing p, $(\mathcal{U}, \phi)$. Note that we assume that $\phi(p) = 0$ by translation.

FINISH

Tangent Bundle

Definition: *The tangent bundle of $\mathcal{M}$ is*

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$$

Theorem. *$T\mathcal{M}$ is a differentiable manifold of dimension $2 \cdot$dim($\mathcal{M}$)*

Vector Bundles

Definition: *A differentiable vector bundle $\mathcal{B}$ over a differentiable manifold $\mathcal{M}$ is a differentiable manifold with the properties:*

- *There is a differentiable surjection $\pi : \mathcal{B} \to \mathcal{M}$ (projection of the bundle)*
- *$\forall p \in \mathcal{M}$ $\mathcal{B}_p = \pi^{-1}(p) \cong V$ where V is a fixed finite dimensional vector space*
- *There is an open cover $\{U_\alpha\}$ of $\mathcal{M}$ such that $\forall\alpha$ $\exists\phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to U_\alpha \times V$ a diffeomorphism*

  In particular we should have in mind $\phi_\alpha(v) = (\pi(v), \omega \circ v)$ where ω is the map given by the second condition. i.e. $\phi|_{\pi^{-1}(q)} = \omega$

  This is called the local triviality condition.

Lemma. *(In the real case)*

$$dim(\mathcal{B}) = dim(V) + dim(\mathcal{M})$$

Trivial Bundles

Definition: *The vector bundle (with the obvious structure) $\mathcal{M} \times \mathbb{R}^n$ is called the trivial (n-plane) bundle of $\mathcal{M}$*

Definition: *Two vector bundles, $\mathcal{B}_1, \mathcal{B}_2$, over $\mathcal{M}$ are equivilent $\iff$ $\exists h : \mathcal{B}_1 \to \mathcal{B}_2$ a homeomorphism such that $\forall p \in \mathcal{M}\ \pi_1^{-1}(p) \cong \pi_2^{-1}(p)$ by h.*
i.e. h is an isomorphism on the fibres.

Definition: *A vector bundle $\mathcal{B}$ over $\mathcal{M}$ is trivial if it is equivalent to the trivial bundle over $\mathcal{M}$ (for some n)*

Orientability

Consider a vector space V with two bases $(v_1, ..., v_n), (v_1', ..., v_n')$ and a change of basis matrix A (i.e. an isomorphism $A : V \to V$)

Definition: *Two bases are equally oriented iff $detA > 0$*

This is an equivilence relation dividing bases into two classes. A choice of class is called an orientation for V. The orientation given by the standard basis on V is the standard orientation.

Definition: *V and W n dimensional vector spaces with respective orientations μ, ν. An isomorphism $A : V \to W$ is orientation preserving iff*

$$(v_1, ..., v_n) \text{ has orientation } \mu \implies (Av_1, ..., Av_n) \text{ has orientation } \nu$$

Lemma. *An isomorphism of the trivial bundle (of dimension n) over $\mathcal{M}$ is either orientation preserving or reversing.*

Definition: *An orientation of a vector bundle $\mathcal{B}$ over $\mathcal{M}$ is a choice of orientation μ_p for $\mathcal{B}_p$ such that $\forall U \subseteq \mathcal{M}$ open, if $\phi : \pi^{-1}(U) \to U \times V$ is an equivalence and $U \times V$ has standard orientation then $\forall q \in U,\ \phi_q = \phi|_{\mathcal{B}_q}$*

If the compatability condition is satisfied for any equivilence it is satisfied for all of them.

Proof.

Definition: *A vector bundle is called orientable if it has an orientation;*

An oriented bundle is a vector bundle together with an orientation.

For an oriented vector bundle the isomorphisms on the fibres to the fixed vector spaces are either all orientation preserving or all reversing.

Definition: *A manifold $\mathcal{M}$ is orientable if $T\mathcal{M}$ is an orientable bundle;*

An oriented manifold is a manifold together with an orientation.

Sections

Definition: *A continously differentiable section of a vector bundle $\mathcal{B}$ over a manifold differentiable manifold $\mathcal{M}$ is a continuously differentiable map*

$$\sigma : \mathcal{M} \to \mathcal{B}; \quad \pi \circ \sigma = id_{\mathcal{M}}$$

For a bundle $\mathcal{B}$ over $\mathcal{M}$ we have each fibre isomorphic to $\mathbb{R}^m$ and an open cover together with diffeomorphisms, $(\mathcal{U}_\alpha, \omega_\alpha)$ (the local trivilisation).

Definition: *We have the local basis sections $\sigma_{\alpha,b}, b = 1, ..., m$ (m and α from above).*

$$\sigma_{\alpha,b} = (\omega_\alpha|_p)^{-1} \cdot e_b$$

For e_b the standard basis for $\mathbb{R}^m$.

These are called basis sections because $\forall p \in \mathcal{U}_\alpha$ $\{\sigma_{\alpha,b}(p) : b = 1, ..., m\}$ forms a basis for $\mathcal{B}_p$. Thus sections of $\mathcal{B}$ over $\mathcal{M}$ can be expanded over $\mathcal{U}_\alpha$ as

$$\psi = \sum_{b=1}^{m} \psi_\alpha^b \sigma_{\alpha,b}$$

For some ψ_α^b real valued functions.

6.3.1 Sections of the Tangent Bundle

Definition: *A continously differentiable section X of $T\mathcal{M}$ is a continuosly differentiable vectorfield on $\mathcal{M}$.*

An alternative Definition is as follows: A C^∞ vectorfield on $\mathcal{M}$ is a linear operator $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ satisfying the Leibniz rule

$$X(fg) = f(Xg) + g(Xf)$$

Wikipedia has satisfying the leibniz rule as the Definition not this huh? Xf(p) = X(p)(f)??
X(p)????
Is satisfying the leibniz rule a condition on the Definition or a consequence??

6.3.2 Integral Curves

Definition: *An integral curve of a continously differentiable vector field V (on $\mathcal{M}$) through a point $p \in \mathcal{M}$ is a continously differentiable curve $\gamma : (-1, 1) \to \mathcal{M}$ such that $\gamma(0) = p$ and $t \in (-1, 1), \dot{\gamma}(t) = V(\gamma(t))$.*
Where $\dot{\gamma}(t)(f) = \frac{d}{dz}(f \circ \gamma)(z)|_{z=t}$.

The problem of finding integral curves to a vector field is the problem of finding solutions to ODEs, namely

$$\frac{d(\phi \circ \gamma)_i}{dt}(t) = (V_i \circ \phi^{-1})(\phi \circ \gamma(t)); \quad i = 1, ..., n$$

with initial conditions $(\phi \circ \gamma)(0) = \phi(p)$, and where ϕ is the homeomorphism from the local chart that contains p.

Dont think its this simple because the image of γ neednt be in a single chart so you need to compartmentalise a bit.

Definition: *A vector field on $\mathcal{M}$ is complete if each integral curve exists for all time (has a global solution)*

Definition: *A family of differomorphisms parametrised by $t \in \mathbb{R}$ having the property that*

$$\phi_0 = id_{\mathcal{M}}, \quad \phi_s \circ \phi_t = \phi_{s+t}, \quad t, s \in \mathbb{R}$$

is calle a 1 parameter group of diffeomorphisms of $\mathcal{M}$.

Theorem. *If $\mathcal{M}$ is a compact manifold then any vector field is complete.*

Proof. Let V be a vector field of $\mathcal{M}$, a compact manifold. For each $p \in \mathcal{M}$ there is an open set $p \in \mathcal{U}_p$, when collected these form a cover. By local existence and uniqueness for each $q \in \mathcal{U}_p$ there is an ϵ_p such that the integral curve (with initial condition $\gamma_q(0) = q$) exists on $t \in [-\epsilon_p, \epsilon_p]$. (The time of the solution is dependent only on the open set in which it is taken, called local well posedness).

Now because $\mathcal{M}$ is compact we can take a finite subcover $\{\mathcal{U}_{p_i}\}_{i=1}^n$. Now we can take $\epsilon = min(\epsilon_{p_1}, ..., \epsilon_{p_n})$ and this gives that (with initial condition) the integral curve at every $q \in \mathcal{M}$ exists for all $t \in [-\epsilon, \epsilon]$. Becuase of this we have a collection of well defined maps (one for each $t \in [-\epsilon, \epsilon]$)

$$\phi_t(q) = \gamma_q(t) : \mathcal{M} \to \mathcal{M}$$

We can extend ϕ to all real values of t by decomposing $t = [\frac{t}{\epsilon}]\epsilon + s_t$, where $[\frac{t}{\epsilon}]$ is the integer part and s is the decimal part of t.

Then we define

$$\phi_{[\frac{t}{\epsilon}]\epsilon} = \begin{cases} \phi_\epsilon \circ ... \circ \phi_\epsilon & [\frac{t}{\epsilon}] \text{ fold composition when } [\frac{t}{\epsilon}] > 0 \\ \phi_{-\epsilon} \circ ... \circ \phi_{-\epsilon} & [\frac{t}{\epsilon}] \text{ fold composition when } [\frac{t}{\epsilon}] < 0 \end{cases}$$

Finally we can define $\phi_t = \phi_{[\frac{t}{\epsilon}]\epsilon} \circ \phi_s$ for all $t \in \mathbb{R}$ (note that because $s \in [0, \epsilon]$ by construction ϕ_s is already well definied). The last thing to check is that $\phi_a \circ \phi_b = \phi_{a+b}$ which is true by unproved ODE black magic.

Now we have for any point and time a well defined integral curve and so V is complete.

Note that any complete vector field generates a 1-parameter group $\{\phi_t\}$ of diffeomorphisms that is called the flow.

$$\phi_t(q) = \gamma_q(t)$$

Where γ_q is the unique integral curve satisfying the initial condition $\gamma_q(0) = q$ (guaranteed unique by ODE theory), moreover γ_q is defined for all t becuase the vector field is complete.

Cotangent Bundle

6.4.1 Differential

Definition: *The differential of a C^∞ map $\psi : \mathcal{M} \to \mathcal{N}$ at a point $p \in \mathcal{M}$ is*

$$d\psi(p) : T_p\mathcal{M} \to T_{\psi(p)}\mathcal{N}; \quad u \mapsto u(- \circ \psi)$$

What about later when we dont use a p are we "implicitly" using one or is there a notion of taking the differential on the whole space

Cotangent Bundle

Definition: *The cotangent bundle of $\mathcal{M}$ is*

$$T^*\mathcal{M} = (T\mathcal{M})^* = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$$

Sections

Recall the local basis sections of $\mathcal{B}$ over $\mathcal{U} \subseteq \mathcal{M}$, $\iota_a : a = 1, ..., n$. The dual local basis sections of $\mathcal{B}^*$ over $\mathcal{U}$ are $\iota^{*a} : a = 1, ..., n$ defined by $\iota^{*a}\iota_b = \delta^a_b$ (Kroneckers delta).

Definition: *A continuously differentiable 1-form on $\mathcal{M}$ is a continuously differentiable section of $T^*\mathcal{M}$*

we use the following notation:

- $\mathscr{X}^{\infty}(\mathcal{M})$ is the set of smooth vector fields on $\mathcal{M}$
- $\Omega_1^{\infty}(\mathcal{M})$ is the set of smooth 1-forms on $\mathcal{M}$

Pull backs and Pushforwards

Apply to covariant objects? Covarient is that when we change coordinates of the problem we change the answer by the same matrix? More generally?
Is there a general formulation for any covarient object?

Definition: *The pullback of a cd function, f, on $\mathcal{N}$ by a cd map $\phi : \mathcal{M} \to \mathcal{N}$ is the cd function on $\mathcal{M}$ given by $\phi^* f = f \circ \phi$*

Definition: *The pullback of a 1-form, θ, on $\mathcal{N}$ by a cd map $\phi : \mathcal{M} \to \mathcal{N}$ is given by*

$$(\phi^*\theta)(p)(-) = \theta(\phi(p))(d\phi(-)) : T_p\mathcal{M} \to \mathbb{R}$$

The pull back of a one form on $\mathcal{N}$ is a one form on $\mathcal{M}$.

Similar question for contravarient objects?

Definition: *The push forward of a vector field, V, on $\mathcal{M}$ by a cd injection $\phi : \mathcal{M} \to \mathcal{N}$ is $\phi_* V = (d\phi \cdot V)(\phi^{-1}(-))$ which is definied on the domain $\phi(\mathcal{M})$. (injectivity here is used to take the "inverse" of an object (it is assumed in the image so we only use injectivity to make the inverse unique))*

i.e. for $n \in \mathcal{N}$

$$(d\phi \cdot V)(\phi^{-1}(n)) = d\phi(\phi^{-1}(n)) \cdot V(\phi^{-1}(n)) = V(\phi^{-1}(n))(- \circ \phi) \in T_p\mathcal{N}$$

So the pushforward of a vector field on $\mathcal{M}$ is a vector field $\mathcal{N}$

Integral Curves

Theorem. *Let V be a complete vector field and ψ_t the one parameter group of diffeomorphisms that it generates then*

$$\phi_* V = V \iff \forall t \in \mathbb{R}, \ \ \psi_t \circ \phi = \phi \circ \psi_t$$

Proof. If γ_p is an integral curve of V through p then $\phi \circ \gamma_p$ is the integral curve of $\phi_* V$ through $\phi(p)$. If V is complete we also see that V generates a one parameter family of diffeomorphism

$$\psi_t(p) = \gamma_p(t)$$

Then these integral curves then generate integral curves of $\phi_* V$ which we call

$$\varphi_t(p) = (\phi \circ \gamma_p)(t) = (\phi \circ \psi_t)(p)$$

Thus we conclude that $\varphi_t \circ \phi = \phi \circ \psi_t$.

In other words the one parameter family of diffeomorphisms generated by $\phi_* V$ is obtained by from the one parameter family generated by V via conjugation by ϕ. So we conclude

$$\phi_* V = V \iff \psi_t = \phi \circ \psi_t \circ \phi \iff \psi_t \circ \phi = \phi \circ \psi_t$$

(Only if because the integral curves also determine the vector field)

Lie Algebra

Lie Derivatives

Let X be a complete vector field on $\mathcal{M}$ that generates the one parameter group of diffeomorphisms ϕ_t.

Definition: *Let f be a function on $\mathcal{M}$, the Lie derivative of f is*

$$\mathcal{L}_X f = \frac{d}{dt}\phi^* f|_{t=0}$$

Definition: *Let θ be a 1-form on $\mathcal{M}$, then the Lie derivative of θ is*

$$\mathcal{L}_X \theta = \frac{d}{dt}\phi^* \theta|_{t=0}$$

Definition: *Let Y be a vector field on $\mathcal{M}$ then the Lie derivative of Y with respect to X is*

$$\mathcal{L}_X Y = \frac{d}{dt}\phi_{-t*} Y|_{t=0}$$

Lemma.

$$\mathcal{L}_X f = Xf$$

Proof. f is a function so we can write $\phi_t^* f = f \circ \phi_t$ thus

$$\mathcal{L}_X(p) = \frac{d}{dt}\phi_t^* f|_{t=0}(p) = \frac{d}{dt}(f \circ \phi_t)|_{t=0}(p) = \frac{d}{dt}(f \circ \gamma_{(-)}(t))|_{t=0}(p) = X(p)(f)$$

where $\gamma_p(t)$ is the integral curve that defines the one parameter family of diffeomorphisms, and hence the last step follows becuase by Definition of integral curve $\frac{d}{dt}\gamma_p(t) = V(\gamma_p(t))$ and $\gamma_p(0) = p$.

Theorem.

$$\mathcal{L}_X Y = [X, Y]$$

Proof. From the Definition for any $f \in C^\infty(\mathcal{M})$ and $p \in \mathcal{M}$

$$(\mathcal{L}_X Y)(p)(f) = \lim_{t\to 0}(\frac{\phi_{-t*}Y - Y}{t})(p)(f) = \lim_{t\to 0}\left[(\frac{\phi_{-t*}Y - Y}{t})(p)(f))\right]$$

Simplifying the term in the limit leads to

$$\begin{aligned}(\frac{\phi_{-t*}Y - Y}{t})(p)(f) &= (\frac{(d\phi_{-t} \cdot V)(\phi_{-t}^{-1}(-)) - Y}{t})(p)(f) \\ &= (\frac{Y(\phi_t(p))(- \circ \phi_{-t}) - Y(p)}{t})(f) \\ &= \frac{Y(\phi_t(p))(f \circ \phi_{-t}) - Y(p)(f)}{t}\end{aligned}$$

We can write $f \circ \phi_{-t} - f$ in terms of the mean value of $\frac{d}{dt'} f \circ \phi_{-t}$ on $[0, 1]$

What exactly is the mean here this is wild, do I really need to be able to prove this I think no.

Lemma.

$$(\mathcal{L}_X \theta) \cdot Y = X(\theta \cdot Y) - \theta \cdot [X, Y]$$

Proof.

Lemma.

$$(\mathcal{L}_X\theta)\cdot(fY) = f(\mathcal{L}_X\theta)\cdot Y$$

Proof.

NOT SURE IF I SHOULD BOTHER WITH THE PROOFS OF ABOVE?

Flows

Theorem. *Let X and Y be complete vector fields on $\mathcal{M}$ that generate the 1-parameter groups ϕ_t and ψ_s respectively.*

$$[X, Y] = 0 \implies \forall s, t \in \mathbb{R} \;\; \phi_t \circ \psi_s = \psi_s \circ \phi_t$$

Proof.

Theorem. *Let χ_{st} be the one parameter group generated by the vecotr field [X, Y] (X and Y as above). Then*

$$\psi_{-s} \circ \phi_{-t} \circ \psi_s \circ \phi_t = \chi_{st} + O((s+t)^3)$$

Proof.

Exterior Derivatives

Let V be a vector space, then we denote the space of anti-symmetric bilinear forms on V by $\Lambda_2(V)$

Then we define a bundle

$$\Lambda_2\mathcal{M} = \bigcup_{p\in\mathcal{M}} \Lambda_2(T_p\mathcal{M})$$

Definition: *A two form on $\mathcal{M}$ is a continuously differentiable section of $\Lambda_2(\mathcal{M})$*

Definition: *Let $\alpha, \beta \in V^*$ then we define $\alpha \wedge \beta \in \Lambda_2(V)$ (the outer product of α and β) as*

$$(u, v) \mapsto (\alpha\cdot u)(\beta\cdot v) - (\alpha\cdot v)(\beta\cdot u)$$

Definition: *Given a 1-form θ on $\mathcal{M}$ the 2-form $d\theta$ (the exterior derivative of θ) is defined as a map on $\mathscr{X}^2(\mathcal{M})$ by*

$$(X, Y) \mapsto X(\theta\cdot Y) - Y(\theta\cdot X) - \theta\cdot[X, Y]$$

Lemma. *For θ a 1-form*

$$\mathscr{L}_X(\theta) = d\theta\cdot(X, -) + d(\theta\cdot X)$$

Lie Groups

Definition: *A Lie group G is a group which is also a differentiable manifold and the group operations of inversion and multiplication are continuously differentiable*

We have two distinguished groups of automorphisms on G, namely

- Right Multiplication: $\{r_a : a \in G\}$, where $r_a(b) = ba$
- Left Multiplication: $\{l_a : a \in G\}$, where $l_a(b) = ab$

Definition: *The Lie algebra of a Lie group is*

$$\mathscr{G} = \{X \in \mathscr{X}^{\infty}(\mathcal{M}) : (l_a)_* X = X, a \in G\}$$

Lemma.

$$X, Y \in \mathscr{G} \implies [X, Y] \in \mathscr{G}$$

Proof. We use the fact that for any diffeomorphism $\phi : G \to G$

$$\phi_*[X, Y] = [\phi_* X, \phi_* Y]$$

then use that $\ell_a, x \mapsto ax$ is a diffeomorphism giving

$$(\ell_a)_*[X, Y] = [(\ell_a)_* X, (\ell_a)_* Y] = [X, Y]$$

Because X and Y are in the Lie Algebra they are left invariant. Thus $[X, Y]$ is left invariant and therefore also in the Lie algebra.

Construction of Left Invariant Vector Fields

Left invarient vector fields are defined by their value at the origin

$$X(a) = dl_a \cdot X(e)$$

Where $a \in G$ and e is the neutral element of G.

Proof.

We can define the evaluation map

$$\epsilon_p : \mathscr{X}^{\infty}(\mathcal{M}) \to T_p\mathcal{M}, X \mapsto X(p)$$

Notice that ϵ_p restricted to $\mathscr{G}$ is an isomorphism onto T_pG.

8.5.1 Integral Curves

Given $X \in \mathscr{G}$ define ${}^{(X)}\gamma : \mathbb{R} \to G, t \mapsto^{(X)} a_t$ to be the integral curve of X through e.

Lemma. $\{{}^{(X)}a_t : t \in \mathbb{R}\}$ *is a one parameter subgroup of G*

Proof.

Lemma. *For any $b \in G$ the integral curve of X through b is*

$$t \mapsto b^{(X)}a_t$$

Proof.

Lemma. *Left invariant vector fields generate right multiplication*

Proof.

Metrics

A symmetric bilinear form on a real vector space V is also called a quadratic form. We denote the set of all such functions $S_2(V)$.

Definition: *For $\alpha, \beta \in V^*$ we define $\alpha \otimes \beta$, a bilinear form on V, (the tensor product) as*

$$\alpha \otimes \beta(u, v) = \alpha(u)\beta(v)$$

We can symmetrise the tensor by taking the symmetric tensor product of α and β which is $\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$

For a vector bundle $\mathcal{B}$ over a manifold $\mathcal{M}$ we can define the manifold of quadratic forms as

$$S_2(\mathcal{B}, \mathcal{M}) = \bigcup_{p \in \mathcal{M}} S_2(\mathcal{B}_p)$$

Definition: *We define the space of innerproducts on V as $S_2^+(V)$ the positive definite symetric bilinear forms. i.e. $\forall h \in S_2^+(V), h(v, v) \geq 0$ and $h(v, v) = 0 \iff v = 0$*

We know that linear functionals form a vector space so we need to consider the fibres and in particular S2V here as being vector spaces in their own right, i.e. they have a zero element.
In our earlier constructions I think we also use this.
In particular the zero here (v = 0) is a zero function so we are using that = the function space is a vector space

Lemma.

$S_2^+(V)$ is an open positive convex cone of $S_2(V)$

This means it is convex, open and $h \in S_2^+(V), \lambda > 0 \implies \lambda h \in S_2^+(V)$

Proof.

There was a whole discussion on what precissely is the topology on this space, its not really well definied I guess but think about it again and try to trace the roots of the issue.

Definition: *A metric m on a real vector bundle $\mathcal{B}$ is a continuously differentiable section of*

$$S_2^+(\mathcal{B}.\mathcal{M}) = \bigcup_{p \in \mathcal{M}} S_2^+(\mathcal{B}_p)$$

This is an open subbundle of $S_2(\mathcal{B}.\mathcal{M})$ Which is a bundle over $\mathcal{M}$.

We can think of m as a continuously differentiable assignment of an innerproduct to each fibre over $\mathcal{M}$.

When $\mathcal{B} = T\mathcal{M}$ we call the metric a Riemannian metric.

Note that we have defined sections as continuously differentiable, but is there a more general notion of a section that is just any right inverse to the projection. Then for this type of seciton we always know that they exist. The tangent bundle is a manifold of dimension higher than the original manifold, in particular its non-empty and so for a general seciton one must always exist.
Next question is when do innerproduct on the fibres exist? Well the fibres are always isomorphic to $\mathbb{R}$n and so I susspect that they always have a family of possible innerproducts... Then every smooth manifold must have a riemannian metric I suppose.

So what can go wrong specifically if I weaken the smoothness condition...: More functions will be included in the tangent bundle right, because in particular the smooth maps will still satisfy the condition (although one must be careful because there may be fewer smooth maps, however there should still be zero). (all fine)
So we might now have an isomorphism between the fibres and a finite dimensional vector space I suppose. Thus there might be no innerproducts from that? But is it possible that the fibre over an arbitrary space is totally out of control? What does the whole space of functions on M that obey the leibniz rule look like, is the set of continuous functions on M to $\mathbb{R}$controlled in anyway, for some spaces the set of continuous functions is a vector space right, if it were would we be able to say anything about it? It is a vector space over $\mathbb{R}$.
What about the functionals on THAT vector space? Well its a vector space that is a subset of all functions, so the dimension of the vector space must be less than or equal to 2^κ but strictly greater than κ where κ is the cardinal of the basis of $Cts(\mathcal{M}, \mathbb{R})$ (because its a metric topology locally homeomorphic to $\mathbb{R}$n I think this is true (greater than statement?)). So its a vector space and we know the dimension.
Are the leibiz obeying functionals a subspace. They contain zero, closed under multiplication and addition? Should be by the same argument that in the smooth case the tangent space is a vector space right?
So taking the symmetric bilinear forms still makes sense. In particular for any two elements in this space there is the symmetric tensor product. So all we need is that there is a positive definite one for each fibre... Cant think of a proof but it seems inconcievable that this wouldnt exist?

Ok so the tangent space will not form a bundle because the fibres wont be isomorphic to a single vector space and so the notion of a section to the innerproduct collection begins to fall apart.
I dont want to delete these ramblings because they took a long time

Arc Length

Note that we define arc length and volume only for the tangent bundle.

Definition: *The arc length of a curve $\gamma : [a, b] \to \mathcal{M}$ is*

$$L(\gamma[a, b]) = \int_{[a,b]} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt$$

Where $g_{\gamma(t)}$ is the Riemannian metric g evaluated at $\gamma(t)$, $\dot{\gamma}$ is

not defined in our class; Potentially the pushford of the derivative as an element of the tangent space of [a,b] considered as a manifold

Equivilent characterisation of the tangent bundle as a set of equivilence classes of paths???

Lemma. *This integral is independent of the parametrization of γ*

Proof.

Volume

Given a vector space, V, with a basis, $e_1, ..., e_n$ we want to assign volumes. Recall that we can also orient the vector space.

Definition: *A volume form ω on an oriented vector space V is a totally antisymetric n-linear form, where $dim(V) = n$ such that for any positive basis $(e_1, ..., e_n)$ of V*

$$\omega(e_1, ..., e_n) > 0$$

If V is also an inner-product space then there is a unique volume form ω corrosponding to the innerproduct determined by sending

$$\omega(e_1, ..., e_n) = 1$$

9.2.1 Volume Form on Orientable Riemannian Manifold

$(\mathcal{M}, g)$ an m dimensional oriented manifold. At each $p \in \mathcal{M}$ the innerproduct g_p and the orientation on $T_p\mathcal{M}$ determine a volume form ω_p on $T_p\Updownarrow$.

$\omega : \mathcal{M} \to \Lambda_m\mathcal{M}$ is continuously differentiable is g is, and we call it the volume form of $(\mathcal{M}, g)$. We denote ω here by $d\mu_g$.

Lemma. *In any chart $(\mathcal{U}, \phi)$ the volume form $d\mu_g$ is given by*

$$d\mu_g = \sqrt{det(g)}dx_1 \wedge ... \wedge dx_m$$

Proof.

Definition: *Let $\mathcal{D} \subseteq \mathcal{U}$ then we define the volume of $\mathcal{D}$ to be*

$$Vol(\mathcal{D}) = \int_{\mathcal{D}} \omega = \int_{\phi(\mathcal{D})} \omega(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}) \circ \phi^{-1} dx_1...dx_m$$

Proof. That this is well defined:

Partition of Unity

We dont want $\mathcal{D}$ to have to be contained in the domain of a chart.

Let $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\beta) : \alpha \in I\}$ be an atalas for $\mathcal{M}$. Assume that $\mathcal{M}$ is paracompact, namely that $\forall p \in \mathcal{M} \exists \mathcal{V}_p$ an open neighbourhood which intersects only finitely many of the $\mathcal{U}_\alpha$. Realtive to $\mathcal{A}$ we can then construct a partition of unity. That is a collection of functions $\{f_\alpha : \alpha \in I\}$ satisfying

- $\forall\alpha$ f_α is nonnegative, smooth and supported in $\mathcal{U}_\alpha$
- $\sum_\alpha f_\alpha = 1$

Given any domain $\mathcal{D}$ we can define

$$Vol(\mathcal{D}) = \sum_\alpha \int_{\mathcal{D}} f_\alpha \omega$$

Note that this is always positive however it may diverge.

Metric Spaces

Topological Spaces

Equivilence Relations

Bump Functions and Cut-offs

Bumps

A bump function, ρ, is a smooth function on $\mathbb{R}^n$ which is spherically symetric

$$|x'| = |x| \implies \rho(x) = \rho(x')$$

which is non-increasing in $|x|$

$$|x'| > |x| \implies \rho(x') \leq \rho(x)$$

and $\rho(0) = 1,\ \forall x \notin B_1(0),\ \rho(x) = 0$

Cut Off's

A cutoff function can be constructed from bump functions.

Begin by constructing smooth functions, with parameters $a, b \in \mathbb{R}$, denote such a function $\sigma_{a,b}$. We construct σ to have the following properties:

- non-decreasing smooth function
- Have range $[0, 1]$
- $\sigma_{a,b}(t) = 0$ for $t \leq b - a$
- $\sigma_{a,b}(t) = 1$ for $t \geq b + a$

We can now define a cutoff function for given parameters $0 < \epsilon_1 < \epsilon_2$ as a smooth function on $\mathbb{R}^n$ given by

$$\eta_{\epsilon_1,\epsilon_2}(x) = 1 - \sigma_{a,b}(|x|)$$

Where a and b are defied by $b - a = \epsilon_1, b + a = \epsilon_2$.

This cutoff function has the properties that it is spherically symmetric, nonincreasing in $|x|$, has a range $[0, 1]$, $\eta(\overline{B_{\epsilon_1}}(0)) = 1$ and $\eta(\mathbb{R}^n \setminus B_{\epsilon_2}(0)) = 0$.

ODEs

Theorem. *Local Existence: Let $\mathcal{V}$ be an open set in $\mathbb{R}^n$ and V a C^1 vector field on $\mathcal{V}$ then for any $x_0 \in \mathcal{V} \exists r > 0$ and an interval $I(x_0)$ such that for any $y \in B_r(x_0)$ the ODE problem*

$$\frac{dx}{dt} = V(x); \quad x(0) = y$$

has a solution, $x(t; y)$ defined on $I(x_0)$ which is C^2 in t at each y and C^1 in y at each t.

Theorem. *Global Uniqueness: For every $y \in \mathcal{V}$ the problem has a unique maximal solution defined on a unique maximal open interval $0 \in J(y)$.*

Definition: *When $J(y) = \mathbb{R}$ we say we have a global solution for initial condition y.*

Theorem. *Either $supJ(y) = \infty$ or there is no compact subset $K \in \mathcal{V}$ such that the motion in $[0, supJ(y))$ is contained in K.*

The motion being $x(t), t \in [0, supJ(y))$

Theorem. *If V is a C^1 vector field on $\mathbb{R}^n$ and there are positive constants A, B such that*

$$|V(x)| \leq A|x| + B$$

then there is a global solution for all initial conditions $y \in \mathbb{R}^n$

Proof.

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